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Two remarks on the subadditivity inequalities in von Neumann algebras [1]

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Taking into account a developed theory of operator monotone and operator convex functions (see e.g. [**1**–**3**]) it appears interesting to study operator subadditive functions. We do this within the context of von Neumann algebras though the main result is essentially a statement on 2×2-matrices.

In what follows we suppose that M is a von Neumann algebra and $\phi : [0,\infty)\to\mathbb{R}$ is a Borel measurable function bounded on bounded subsets of $[0,\infty)$. We say that ϕ is *operator subadditive with respect to M* or briefly *M-subadditive* if $\phi(a+b)\leq\phi(a)+\phi(b)$ for every pair a, b of positive operators from M.

EXAMPLES. It is easy to see that the following functions on $[0,\infty)$ are M-subadditive for any M:

1) $\phi(t)=\alpha t+\beta$ ($\alpha\in\mathbb{R}$, $\beta\in\mathbb{R}^+$);

2) $\phi(t)=1/(\alpha t+1)$ ($\alpha\in\mathbb{R}^+$);

3) an arbitrary function ϕ satisfying $\alpha\leq\phi(t)\leq 2\alpha$ for some $\alpha\in\mathbb{R}^+$.

Subadditive real functions used in analysis often satisfy $\phi(0)=0$. The following theorem shows that the class of operator subadditive functions satisfying this condition uses to be very small.

THEOREM 1. *Let M be a von Neumann algebra and let there exist a function $\phi:[0,\infty)\to\mathbb{R}$ such that $\phi(0)=0$, ϕ is M-subadditive, and ϕ is not of the form $\phi(t)=\alpha t$ with $\alpha\in\mathbb{R}$. Then M is commutative.*

Proof. Let M be noncommutative. Suppose ϕ is M-subadditive and $\phi(0)=0$. We will show that ϕ has to be of the form $\phi(t)=\alpha t$ for some $\alpha\in\mathbb{R}$.

Since M is noncommutative, it is easy to check that there exist two equivalent and mutually orthogonal nonzero projections in M, i.e., there exists a nonzero partial isometry $v\in M$ such that the projections $p=v^*v$ and $q=vv^*$ are mutually orthogonal (see e.g. [4]). Take positive reals ϵ, δ such that $\epsilon\leq\delta$ and consider the pair of operators:

$$a=\epsilon p+\sqrt{\epsilon(\delta-\epsilon)}\,v+\sqrt{\epsilon(\delta-\epsilon)}\,v^*+(\delta-\epsilon)\,q,$$

$$b=\epsilon p-\sqrt{\epsilon(\delta-\epsilon)}\,v-\sqrt{\epsilon(\delta-\epsilon)}\,v^*+(\delta-\epsilon)\,q.$$

Observe that a and b are positive scalar multiples of projections and a straightforward computation shows that $\phi(a+b)=\phi(2\epsilon)p+\phi(2(\delta-\epsilon))q$, $\phi(a)=(\phi(\delta)/\delta)a$, $\phi(b)=(\phi(\delta)/\delta)b$. Whence, after multiplying the inequality $\phi(a+b)\leq\phi(a)+\phi(b)$ by p from the left and the right we obtain $\phi(2\epsilon)p\leq(2\epsilon\phi(\delta)/\delta)p$. Hence, $\phi(2\epsilon)/2\epsilon\leq\phi(\delta)/\delta$. As the only restriction imposed on ϵ and δ is $0<\epsilon\leq\delta$, it follows that $\phi(t)/t$ is a constant, say α, on $(0,\infty)$. Thus, $\phi(t)=\alpha t$ on $[0,\infty)$.

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By similar arguments, we can also prove the following.

THEOREM 2. *Let τ be a semifinite normal faithful trace on a noncommutative von Neumann algebra M. Let $\phi : [0,\infty) \to \mathbb{R}$ satisfy $\phi(0) = 0$ and*

$$\tau(\phi(a+b)) \leq \tau(\phi(a)) + \tau(\phi(b)) \qquad (*)$$

for every pair a, b of positive operators from M such that both sides of the inequality are well-defined. Then ϕ is concave on $[0,\infty)$.

This theorem may be viewed as a converse to the well-known assertion that, under certain restrictions, the inequality $(*)$ holds if ϕ is supposed to be a concave function on $[0,\infty)$ with $\phi(0) = 0$ (see [5], [6], [7]).

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